



Ensemble phase averaged equations for multiphase flows in porous media. Part 2: A general theory

Duan Z. Zhang *

Theoretical Division, Fluid Dynamics and Solid Mechanics Group, T-3, B216, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

ARTICLE INFO

Article history:

Received 18 February 2009

Received in revised form 3 March 2009

Accepted 11 March 2009

Available online 20 March 2009

Keywords:

Porous media

Multiphase flow

Ensemble phase average

ABSTRACT

Most models for multiphase flows in a porous medium are based on a straightforward extension of Darcy's law, in which each fluid phase is driven by its own pressure gradient. The pressure difference between the phases is thought to be an effect of surface tension and is called capillary pressure. Independent of Darcy's law, for liquid imbibition processes in a porous material, diffusion models are sometime used. In this paper, an ensemble phase averaging technique for continuous multiphase flows is applied to derive averaged equations and to examine the validity of the commonly used models. Closure for the averaged equations is quite complicated for general multiphase flows in a porous material. For flows with a small ratio of the characteristic length of the phase interfaces to the macroscopic length, the closure relations can be simplified significantly by an approximation with a second order error in this length ratio. This approximation reveals the information of the length scale separation obscured during an averaging process and leads to an equation system similar to Darcy's law, but with additional terms. Based on interactions on phase interfaces, relations among closure quantities are studied.

© 2009 Elsevier Ltd. All rights reserved.

1. Introduction

Flow in porous media has been studied extensively because of the important applications in many industries, such as petroleum, chemical and environmental. Most models for flows in a porous medium are based on Darcy's law, in which the velocity is proportional to the pressure gradient in the fluid. Darcy's law was originally obtained for single phase flows in a porous material. For cases of two-phase flows, often Darcy's law is extended by simply applying the law to each phase separately. The average velocity of each phase is calculated with the pressure gradient of the phase. As a consequence of this extension, two pressure fields are introduced for two-phase flows in a porous medium. The introduction of the two pressure fields leaves the equation system unclosed. To close the equation system, Leverett (1941) introduced a concept of capillary pressure and assumed that the pressure difference between the two phases equals to the capillary pressure representing the effect of surface tension. In the original definition, the capillary pressure is a function of the wetting phase saturation in a porous material. Studies (Hillel, 1980; Miller and Noegi, 1985) have found that the pressure difference exhibits hysteresis. The measured pressure difference takes different values during imbibition or drainage processes. To amend the concept of capillary pressure, Hassanizadeh and Gray (1993) introduced an additional dynamic

capillary pressure, which is proportional to the time rate of change in the wetting phase saturation.

Independent of the generalized Darcy's law approach mentioned above, in other fields, such as in hydrology and in the study of drying process in a porous material, the motion of water is modeled directly as a diffusion process with the diffusivity depending on water saturation (Pachepsky et al., 2003; Lockington and Parlange, 2003), independent of the water pressure. The starting point of this approach is based on the work of Washburn (1921), in which the depth of liquid penetration in a capillary tube is found proportional to the square root of time, similar to the fundamental solution of a diffusion equation. Researchers who advocate this diffusion approach over the generalized Darcy's law approach note that the validity of the generalized Darcy's law for multiphase flows has not been rigorously established and may not be a better assumption than the diffusion model (Hilfer, 2006). Liu et al. (1995) took this approach further. In their model, the gradient of the liquid saturation not only drives the viscous motion but also drives the fluid when inertia effects are not negligible. Diffusion models are also found to be limited (Hall, 2007), and "anomalous diffusions" are observed (Küntz and Lavallée, 2001).

The main objectives of the present paper are to study the averaged equations for multiphase flows in a porous medium and to examine the conditions under which the direct generalization of Darcy's law or the diffusion models can be applied. The starting point of this work is the ensemble phase averaged equations for multi-material interactions. The fluids and the solid porous material are treated as deformable materials. Averaged equations are obtained

* Tel.: +1 505 665 4428; fax: +1 505 665 5926.

E-mail address: dzhang@lanl.gov

not only for the fluid phases but also for the solid phase. The ensemble averaging method was initially used to study stress systems in particle suspensions (Batchelor, 1970). Based on the method, the ensemble phase averaging method is developed to derive averaged equations for disperse multiphase flows (Zhang and Prosperetti, 1994, 1997). In this method, the average is performed over all possible flows according to the probability of the flow system. Originally the method is restricted to systems with finite degrees of freedom. Recently (Zhang et al., 2007), this method has been extended for systems with infinite number of degrees of freedom by using abstract probability theory (Ash, 1972; Drew and Passman, 1999). Averaged equations derived strictly from the abstract probability theory, although quite general, are inconvenient for studying interesting physical interactions at different length and time scales, because the averages are calculated by averaging over different flow configurations. In contrast, in this work the abstract probability theory is combined with a small interface approximation to capture the effects of the phase interactions at different length scales. The small interface approximation is an extension of the small particle approximation (Zhang and Prosperetti, 1994, 1997) for disperse two-phase flows. With the small interface or the small particle approximation, closure quantities can be calculated by first integrating over regions (lines or surfaces) in a flow in the ensemble; and then averaging over all possible flows with these regions in specified locations. These regional integrals reveal physical length scales implied in the closure relations. These length scales are intrinsic to the physical system rather than preset as in a volume averaging method. With the small interface approximation, the averaged equations can be simplified, and closure terms can be calculated to study interactions on the phase interfaces.

These calculations yield useful restrictions on the closure relations for multiphase flows in a porous medium. Using this ensemble phase averaging method, we have obtained averaged equations not only for the fluid phases, but also for the solid phase of the porous material. We show that the surface tension force produces a stress acting on the solid phase causing the solid phase to contract during a drying process.

The presentation of this work is divided into two parts as separated papers in this Journal. In Part 1 (Yang et al., 2009), we illustrate the concept of the ensemble phase averaging method through a simple example, the bundle-of-tubes model for a porous material. This paper is Part 2. In this paper we describe a general theory for multiphase flows in porous material. We find that both the generalized Darcy's law and the diffusion model are correct under very limited conditions. Examples are given in Part 1, in which the generalized Darcy's law and the diffusion model are incorrect. If the viscosity of one of the fluid phases is negligible in a two-phase flow in a porous material, the dynamic part of the "capillary pressure" constitutes as a term in the momentum equation for the other fluid phase; however, this "capillary pressure" is not necessarily related to surface tension on the phase interface.

2. Averaged equations

Many averaged equations for flow in a porous medium have been derived using representative volume elements (RVEs) (Pedras and de Lemos, 2001; Breugem and Rees, 2006; Nordbotten et al., 2007, 2008). In this paper, the starting point is the averaged equations derived by using the ensemble phase averaging technique (Zhang and Prosperetti, 1994; Zhang et al., 2007).

In the ensemble phase averaging method, we consider an ensemble of flows and denote a flow belonging to the ensemble as \mathcal{F} . Let $C_i(\mathbf{x}, t, \mathcal{F})$ be the indicator function of phase i , such that $C_i(\mathbf{x}, t, \mathcal{F}) = 1$ if the spatial point \mathbf{x} is occupied by phase i in flow

\mathcal{F} at time t , and $C_i(\mathbf{x}, t, \mathcal{F}) = 0$ otherwise. The volume fraction θ_i of phase i at this point and time is calculated by averaging the values of the indicator functions over all possible flows in the ensemble.

$$\theta_i(\mathbf{x}, t) = \int C_i(\mathbf{x}, t, \mathcal{F}) d\mathcal{P}, \quad (1)$$

where \mathcal{P} is the probability measure (Ash, 1972; Drew and Passman, 1999) defined on the ensemble of the flows, and $\int(\cdot) d\mathcal{P}$ denotes the average over all possible flows in the ensemble according to the probability. The ensemble phase average $\langle q_i \rangle$ for a quantity q_i pertaining to phase i is defined as

$$\langle q_i \rangle(\mathbf{x}, t) = \frac{1}{\theta_i} \int C_i(\mathbf{x}, t, \mathcal{F}) q_i(\mathbf{x}, t, \mathcal{F}) d\mathcal{P}. \quad (2)$$

To focus on interactions on phase interfaces, in the present paper we only consider flows with a constant density for each phase. With this restriction, we do not need to distinguish the average defined in (2) from the Favre average defined as $\bar{q}_i = \langle \rho_i^0 q_i \rangle / \langle \rho_i^0 \rangle$, where ρ_i^0 is the material density of phase i . Extension of this work to compressible flows involves the adoption of the multi-pressure model developed in the paper by Zhang et al. (2007). For incompressible flows the transport equation for the volume fraction can be written as

$$\frac{\partial \theta_i}{\partial t} + \nabla \cdot (\theta_i \mathbf{u}_i) = 0, \quad (3)$$

where \mathbf{u}_i is the average velocity of phase i .

After setting the average stress $\langle \sigma_i \rangle$ of phase i as σ_{Ai} in Eq. (15) in the paper of Zhang et al. (2007), the averaged momentum equation can be written as

$$\frac{\partial}{\partial t} (\rho_i \mathbf{u}_i) + \nabla \cdot (\rho_i \mathbf{u}_i \langle \mathbf{u}_i \rangle) = \theta_i \nabla \cdot \langle \sigma_i \rangle + \nabla \cdot (\theta_i \sigma_i^{Re}) + \mathbf{f}_i + \rho_i \mathbf{g}, \quad (4)$$

where $\rho_i = \theta_i \rho_i^0$ is the macroscopic density of phase i , σ_i^{Re} is the stress due to velocity fluctuations, \mathbf{g} is the body force per unit mass, such as gravity, and

$$\mathbf{f}_i = - \int (\sigma_i - \langle \sigma_i \rangle) \cdot \nabla C_i d\mathcal{P}, \quad (5)$$

is the interfacial force. The integrand in (5) is nonzero only on phase interfaces.

2.1. Surface interactions

To study the effects of phase interactions on interfaces, in this paper, we make the following three assumptions. First, an interface has negligible thickness so that its mass and inertia can be neglected. Second, on the edge of an interface, there is neither a transverse force, the shear force in the direction normal to the interface, nor a shear force parallel to the edge. The only force can exist on an edge of an interface is the surface tension force tangent to the interface and perpendicular to the edge. Third, continuum descriptions are adequate outside the interface region. A surface of a phase may be contacted not only by another phase, but also by phase interfaces, as illustrated in Fig. 1 on a solid surface and in Fig. 2 for more general cases. The intersection of phase interfaces forms a contact line. The first assumption implies that the inertia and the mass of a contact line can also be neglected.

To focus on physics of the multiphase system and to simplify the mathematical derivations, in the main text of this paper we limit our discussion to three-phase systems containing two fluid phases, 1 and 2, and a solid phase s . Extensions to multiphase systems are discussed in Appendix B, using results from Appendix A.

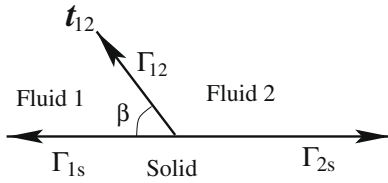


Fig. 1. Intersection of fluid interface with a solid surface.

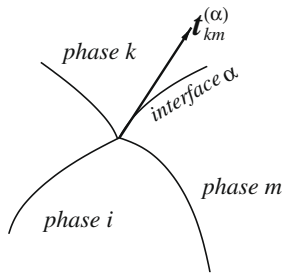


Fig. 2. Illustration of phase interfaces and a tangent vector of the interfaces.

For a contact line on the solid interface as shown in Fig. 1, the balance of surface tension forces in the direction tangent to the solid surface is often written as $\Gamma_{2s} - \Gamma_{1s} - \Gamma_{12} \cos \beta = 0$, where Γ_{1s} and Γ_{2s} are surface tension coefficients on solid surface for both fluids, Γ_{12} is the surface tension coefficient on the fluid interface, and β is the contact angle of fluid 1 on the solid surface. As pointed out by Dussan V (1979), this relation is not sufficient. The force balance in the direction normal to the solid surface requires the solid material to apply a force on the contact line. There are many studies (see the review paper by Dussan V, 1979) on the phase interactions near the contact line on a solid surface. It is not an intention of this paper to discuss this issue. Rather, we only need to assume that the mass associated with a contact line is negligibly small. On the contact line the surface tension force from the fluid interface can be expressed as $\Gamma_{12} \mathbf{t}_{12}$, where \mathbf{t}_{12} is the unit tangent vector of the fluid interface perpendicular to and pointing away from the contact line (see Fig. 1). Under the assumption of negligibly small mass of the contact line, this surface tension force $\Gamma_{12} \mathbf{t}_{12}$ must be balanced by a concentrated force from the solid phase on the contact line. This concentrated force can be a combined contribution from the surface tensions and a force from the solid. For the purpose of this paper, there is no need to know their individual contributions or their precise definitions.

Considering the concentrated force $\Gamma_{12} \mathbf{t}_{12}$ from the fluid interface applied on the contact line on the solid surface, we can write the interfacial force defined in (5) for the solid phase as

$$\mathbf{f}_s = \int \Gamma_{12} \mathbf{t}_{12} \delta_c d\mathcal{P} + \mathbf{F}_{s1} + \mathbf{F}_{s2}, \quad (6)$$

where δ_c is the δ -function that singles out contact lines on the solid surface, and \mathbf{F}_{ij} is the phase interaction force between phases i and j calculated as

$$\mathbf{F}_{ij}(\mathbf{x}, t) = - \int (\boldsymbol{\sigma}_i - \langle \boldsymbol{\sigma}_i \rangle) \cdot (\nabla C_i)_j d\mathcal{P}, \quad (7)$$

with $(\nabla C_i)_j$ denoting ∇C_i restricted on interfaces between phases i and j , excluding the contact lines. The integral in (7) excludes flows in which the point \mathbf{x} is on a contact line at time t because those flows are already accounted for by the first term in (6).

Similarly, we can write the interfacial force defined in (5) for the fluid phases as

$$\mathbf{f}_1 = \mathbf{F}_{12} + \mathbf{F}_{1s}, \quad \mathbf{f}_2 = \mathbf{F}_{21} + \mathbf{F}_{2s}, \quad (8)$$

where \mathbf{F}_{ij} is defined in (7). There is no term similar to the first term in (6) in (8) for the fluid phases because the fluid phases do not apply concentrated forces to contact lines, while the solid phase does.

Since a fluid interface has no mass, the force balance of a fluid interface can be written as

$$\boldsymbol{\sigma}_1 \cdot \mathbf{n}_1 + \boldsymbol{\sigma}_2 \cdot \mathbf{n}_2 + \mathbf{s}_{12} = \mathbf{0}, \quad (9)$$

where \mathbf{n}_i ($i = 1, 2$) is the inward unit normal on surface of phase i , and \mathbf{s}_{12} is the interface traction on the interface. The inward unit normal satisfies $\nabla C_i = \delta_i \mathbf{n}_i$, where δ_i is the δ -function on surface of phase i . Without external forces acting on the interface, the interface traction results from surface tension, and can be written as $\mathbf{s}_{12} = \Gamma_{12} \kappa_{12} \mathbf{n}_1 + \nabla_t \Gamma_{12}$, where $\Gamma_{12} = \Gamma_{21}$ is the surface tension coefficient, κ_{12} is the curvature of the interface, and $\nabla_t = \nabla - \mathbf{n}_1 (\mathbf{n}_1 \cdot \nabla)$ is the gradient operator tangent to the interface. The curvature κ_{12} is defined positive if the interface is convex for phase 1; thus $\kappa_{21} = -\kappa_{12}$. The second term in the expression for \mathbf{s}_{12} is the tangential component of the surface tension (Landau and Lifshitz, 1987).

The force balance on the solid–fluid interface (away from the contact line) can be written as

$$\boldsymbol{\sigma}_s \cdot \mathbf{n}_s + \boldsymbol{\sigma}_i \cdot \mathbf{n}_i = \mathbf{0}, \quad (i = 1 \text{ or } 2). \quad (10)$$

In relation (10) we do not explicitly introduce the surface tension on the solid surface. The response of the solid phase and its surfaces is included in the solid stress $\boldsymbol{\sigma}_s$.

Using (9), we find

$$\mathbf{F}_{12} + \mathbf{F}_{21} = (\langle \boldsymbol{\sigma}_1 \rangle - \langle \boldsymbol{\sigma}_2 \rangle) \cdot \int (\nabla C_1)_2 d\mathcal{P} + \int \mathbf{s}_{12} \delta_{12} d\mathcal{P}, \quad (11)$$

where δ_{12} is the δ -function δ_1 restricted on the interface between fluids 1 and 2 (i.e. $(\nabla C_1)_2 = \mathbf{n}_1 \delta_{12}$). Similarly, using (10), we have

$$\mathbf{F}_{is} + \mathbf{F}_{si} = (\langle \boldsymbol{\sigma}_i \rangle - \langle \boldsymbol{\sigma}_s \rangle) \cdot \int (\nabla C_i)_s d\mathcal{P}. \quad (12)$$

The interface integral $\int (\nabla C_i)_j d\mathcal{P}$ in (11) and (12) represents a morphology effect on multiphase flows. Although it appears to be kinematic in the definition, the dynamics of phase interactions affect phase interfaces, because the shape and normal of the interface are affected by local flow field. For instance, the contact angle, the normal and the shape of the interface are functions of local flow direction. The hysteresis of the contact angle also affects the last term in (11) because \mathbf{s}_{12} is related to the normal of the interface. Since $\nabla C_i = \sum_j (\nabla C_i)_j$ and $\int \nabla C_i d\mathcal{P} = \nabla \theta_i$, models for the interface integral need to satisfy the following relation:

$$\sum_j \int (\nabla C_i)_j d\mathcal{P} = \nabla \theta_i. \quad (13)$$

By using (6) and (11)–(13) and noting that $\nabla(C_2)_1 = -\nabla(C_1)_2$, we can write the interfacial force for the solid phase as

$$\begin{aligned} \mathbf{f}_s = & -\mathbf{F}_{1s} - \mathbf{F}_{2s} + \langle \boldsymbol{\sigma}_s \rangle \cdot \nabla \theta_s + \langle \boldsymbol{\sigma}_1 \rangle \cdot \nabla \theta_1 + \langle \boldsymbol{\sigma}_2 \rangle \cdot \nabla \theta_2 \\ & - (\mathbf{F}_{12} + \mathbf{F}_{21}) + \int \mathbf{s}_{12} \delta_{12} d\mathcal{P} + \int \Gamma_{12} \delta_c \mathbf{t}_{12} d\mathcal{P}. \end{aligned} \quad (14)$$

2.2. Viscous flow in porous solid

We now assume that the fluids are Newtonian with viscosity μ_i ($i = 1, 2$) and $\boldsymbol{\sigma}_i = -p_i \mathbf{I} + \boldsymbol{\tau}_i$, where $\boldsymbol{\tau}_i$ is the viscous stress, p_i is the pressure, and \mathbf{I} is the identity tensor. We further assume that the characteristic length, L_f , of a pore cross section is small compared to the macroscopic length scale L of the problem. The local viscous stress $\boldsymbol{\tau}_i$ can be estimated to be of order $\mu_i (\langle \mathbf{u}_i \rangle - \langle \mathbf{u}_s \rangle) / L_f$ for the fluid of phase i . The flow of fluids in such a porous solid is viscous flow driven by a pressure gradient. From the balance between the pressure gradient force, estimated as $\nabla p_i L_f^2$, and the viscous force

$\tau_i L_f$, we can estimate the pressure gradient to be of order $\mu_i(\langle \mathbf{u}_i \rangle - \langle \mathbf{u}_s \rangle) / L_f^2$; and hence the pressure difference is of order $\mu_i(\langle \mathbf{u}_i \rangle - \langle \mathbf{u}_s \rangle) L / L_f^2$. The ratio of the magnitude of the average viscous stress $\langle \tau_i \rangle$ to the pressure difference in average stress $\langle \sigma_i \rangle$ is then of order L_f / L . If this order of error is allowed in the momentum equations, then the average viscous stress can be neglected in $\langle \sigma_i \rangle$ and $\langle \sigma_i \rangle \approx -\langle p_i \rangle \mathbf{I}$. This approximation is used in almost all models for flows in porous media. For a single phase flow in a rigid porous solid, this approximation is of order L_f^2 / L^2 , one order higher than L_f / L for multiphase flows. This difference is a result of the calculation of the average of the velocity gradient $\langle \nabla \mathbf{u}_i \rangle$, because the average viscous stress can be calculated as $\langle \tau_i \rangle = \mu_i(\langle \nabla \mathbf{u}_i \rangle + \langle \nabla \mathbf{u}_i \rangle^T) / 2$. By taking the spatial derivative of definition (2) with $q_i = \mathbf{u}_i$, we find (Zhang et al., 2007)

$$\langle \nabla \mathbf{u}_i \rangle = \frac{1}{\theta_i} \nabla(\theta_i \langle \mathbf{u}_i \rangle) - \frac{1}{\theta_i} \int \mathbf{u}_i (\nabla C_i)_j d\mathcal{P} - \frac{1}{\theta_i} \int \mathbf{u}_s (\nabla C_i)_s d\mathcal{P}, \quad (15)$$

where we have used the no-slip condition on solid walls in the last term. The first term in the right-hand side of (15) is of order $\langle \mathbf{u}_i \rangle / L$. For a single phase flow in a rigid solid, the second term disappears, and $\mathbf{u}_s = \mathbf{0}$ in the third term for rigid solid walls. For a multiphase flow, however, the second term is of order $\langle \mathbf{u}_i \rangle / L_f$.

According to the analysis above, in the case of a viscous flow in a porous medium, the stress difference $\sigma_i - \langle \sigma_i \rangle$ in (7) linearly depends on the viscosity of the fluid and the relative velocity between the fluid and the solid. We can model the force \mathbf{F}_{is} defined in (7) as proportional to the velocity difference $\langle \mathbf{u}_i \rangle - \langle \mathbf{u}_s \rangle$ and the viscosity of the fluid.

$$\mathbf{F}_{is} = \theta_i C_{is} \mu_i (\langle \mathbf{u}_s \rangle - \langle \mathbf{u}_i \rangle), \quad (16)$$

where C_{is} is the viscous drag coefficient. If the information of the stress distribution at the pore size scale is available, such as in the bundle-of-tubes mode discussed in Part 1 (Yang et al., 2009), the force \mathbf{F}_{is} can be calculated explicitly; and the viscous drag coefficient C_{is} can be determined from its definition (16). For a single phase flow, $1/C_{is}$ is the permeability of the porous material. For multiphase flows, C_{is} is a function of the saturation of the fluid as shown in Part 1 (Yang et al., 2009).

After neglecting inertia terms and using (8), the momentum equation for the fluid phases (4) becomes

$$\theta_i C_{is} \mu_i (\langle \mathbf{u}_i \rangle - \langle \mathbf{u}_s \rangle) = \theta_i \nabla \cdot \langle \sigma_i \rangle + \mathbf{F}_{ij} + \rho_i \mathbf{g}, \quad (17)$$

$$(i = 1, 2, j = 1, 2, j \neq i).$$

Similarly, after neglecting inertia of the solid and using (14), the momentum equation for the solid phase becomes

$$\theta_1 C_{1s} \mu_1 (\langle \mathbf{u}_s \rangle - \langle \mathbf{u}_1 \rangle) + \theta_2 C_{2s} \mu_2 (\langle \mathbf{u}_s \rangle - \langle \mathbf{u}_2 \rangle) = \nabla \cdot (\theta_s \langle \sigma_s \rangle) + \langle \sigma_1 \rangle \cdot \nabla \theta_1 + \langle \sigma_2 \rangle \cdot \nabla \theta_2 - (\mathbf{F}_{12} + \mathbf{F}_{21}) + \int \mathbf{s}_{12} \delta_{12} d\mathcal{P} + \int \Gamma_{12} \delta_c \mathbf{t}_{12} d\mathcal{P} + \rho_s \mathbf{g}. \quad (18)$$

In Part 1 (Yang et al., 2009) of this work we used the concept of ensemble phase average to study the bundle-of-tubes model for multiphase flow. In the following section, we show that the equations obtained in Part 1 are a special case of the more general theory obtained in this paper. We also study the averaged equation of motion for the solid phase in the model. This exercise illustrates the basic character of phase interaction terms introduced in this paper, although this model is overly simplistic for most practical problems. To study practical systems, we need to employ the small interface approximation which is an extension of the small particle approximation originally developed for disperse two-phase flows (Zhang and Prosperetti, 1994, 1997). Because this approximation requires some mathematical derivation, we postpone the discussion to Section 4.

3. Bundle-of-tubes model

As described in Part 1 (Yang et al., 2009), we consider a one-dimensional two-phase flow in a porous material consisting of uniformly distributed capillary tubes with different diameters. Suppose the capillary tubes are parallel to each other and in direction x with length L . Let $P(\phi)$ be the probability density of the diameters, ϕ 's, of the capillary tubes, and β be the contact angle of fluid 1 on the solid wall, measured from the solid wall in contact with fluid 1 to the interface between fluids 1 and 2 as shown in Fig. 1. For small tubes, the curvature radius a of the fluid interface can be calculated as $a = \phi / (2 \cos \beta)$. In this example, there is a reservoir of fluid 1 on the left of the capillary tubes. Initially, the capillary tubes are filled with fluid 2 and are connected to a reservoir of fluid 2 at the right end. At time $t = 0$, the left ends of the capillary tubes are brought into contact with the reservoir of fluid 1 causing the invasion of fluid 1 into the capillary tubes initially filled with fluid 2.

The averaged fluid momentum equations (23) and (24) of Part 1 are special case of (17) with $\langle \mathbf{u}_s \rangle = \mathbf{0}$. In this example, the capillary tubes with different diameters are uniformly distributed and $\int (\nabla C_i)_s d\mathcal{P} = \mathbf{0}$, because at a given x the sum of the unit normal vectors on the solid walls vanishes. Using (13) we have $\int (\nabla C_1)_2 d\mathcal{P} = \int \mathbf{n}_1 \delta_{12} d\mathcal{P} = \nabla \theta_1$. Similarly, $\int (\nabla C_2)_1 d\mathcal{P} = \nabla \theta_2$. In this example as mentioned in Part 1, the diameter ϕ of the tubes containing an interface located at x at time t , is uniquely determined by the interface location and the time, $\phi = \phi(x, t)$. The curvature of the interface can be calculated as $\kappa_{12} = -2/a = -4 \cos \beta / \phi(x, t)$. Using this result, and noting constant surface tension coefficient in the example, we have

$$\int \mathbf{s}_{12} \delta_{12} d\mathcal{P} = \int \kappa_{12} \mathbf{n}_1 \delta_{12} d\mathcal{P} = -\frac{4\Gamma_{12} \cos \beta}{\phi(x, t)} \nabla \theta_1, \quad (19)$$

and (11) becomes

$$\mathbf{F}_{12} + \mathbf{F}_{21} = [\langle p_2 \rangle - \langle p_1 \rangle - 4\Gamma_{12} \cos \beta / \phi(x, t)] \nabla \theta_1, \quad (20)$$

after approximating the stresses by the corresponding pressures. This is Eq. (27) in Part 1.

In a typical model for multiphase flows in porous solid, the response of the solid phase is not studied. To study the solid motion, we need to calculate $\int \Gamma_{12} \delta_c \mathbf{t}_{12} d\mathcal{P}$ in the momentum equation (18) for the solid phase. We again note that the interface location x at time t uniquely determines the diameter $\phi(x, t)$ of the tube, and that each such interface contributes $-\pi \phi(x, t) \Gamma_{12} \cos \beta \mathbf{i}$ to the integral, therefore

$$\int \Gamma_{12} \delta_c \mathbf{t}_{12} d\mathcal{P} = -n^{(12)} \pi \phi \Gamma_{12} \cos \beta \mathbf{i}, \quad (21)$$

where $n^{(12)}(x, t)$ is the number density of the interface at location x at time t , and \mathbf{i} is the unit vector in x -direction. To calculate the number density $n^{(12)}$ we note that the number, $n^{(12)} \Delta x$, of interfaces per unit cross section perpendicular to the x direction in the vicinity Δx of x , can be calculated as $n_A P(\phi) \Delta \phi$, where n_A is the number of tubes per unit cross section area, $\Delta \phi = \phi(x + \Delta x, t) - \phi(x, t)$, and $P(\phi) \Delta \phi$ is the probability of finding the suitable tube diameters. That is

$$n^{(12)}(x, t) = n_A P(\phi(x, t)) \frac{\partial \phi}{\partial x}. \quad (22)$$

Differentiating Eq. (9) in Part 1 with respect to x , we find

$$\nabla \theta_1 = -\frac{1}{4} n_A \pi \phi^2 P(\phi) \frac{\partial \phi}{\partial x} \mathbf{i}. \quad (23)$$

Comparing (22) and (23), we find $n^{(12)}(x, t) \mathbf{i} = -4 / [\pi \phi^2(x, t)] \nabla \theta_1$, and

$$\int \Gamma_{12} \delta_c \mathbf{t}_{12} d\mathcal{P} = \frac{4\Gamma_{12} \cos \beta}{\phi(\mathbf{x}, t)} \nabla \theta_1, \quad (24)$$

after using (21).

In the example of Part 1, only invasion of fluid 1 is considered. In this case, the contact angle β is a constant. If the flow involves invasion of fluid 1 in some of the tubes and withdraw of the fluid in the other tubes, then, because of the contact angle hysteresis, the forces \mathbf{F}_{12} and \mathbf{F}_{21} may become history dependent.

With relations (11), (19) and (24) we find that the two integrals in (18) cancel each other in this example, and the momentum equation becomes

$$\begin{aligned} & \theta_1 C_{1s} \mu_1 (\langle \mathbf{u}_s \rangle - \langle \mathbf{u}_1 \rangle) + \theta_2 C_{2s} \mu_2 (\langle \mathbf{u}_s \rangle - \langle \mathbf{u}_2 \rangle) \\ & = \nabla \cdot (\theta_s \langle \boldsymbol{\sigma}_s \rangle) + 4\Gamma_{12} \cos \beta / \phi \nabla \theta_1 + \rho_s \mathbf{g}. \end{aligned} \quad (25)$$

In writing this equation, the cancellation of the two integrals in (18) is not a coincidence as we shall see in the next section. The second term on the right-hand side implies that the surface tension pulls the solid toward to the invading fluid in the direction parallel to the capillary tube. On the other hand, the surface tension from the fluid interfaces pulls the solid wall toward the center and tends to cause contraction of the tube. This mechanism of possible solid deformation has not been captured in this example. Since the net force perpendicular to the tubes from the surface tension is zero in each tube, the only way for this surface tension effect to appear in the solid momentum equation is through the non-uniformity of the tube distribution. The non-uniformity of tube distribution is of order $1/L$, where L is the macroscopic length over which significant change in the tube density occurs. The length scale of the surface tension force on the contact line is $\bar{\phi}$, the average diameter of the tube. Therefore to consider the possible solid deformation caused by the surface tension, the momentum equation needs to be correct to first order of the length ratio $\bar{\phi}/L$. In the example considered in this section, we have neglected terms of this order because of the invalidity of Poiseuille flow assumption near the fluid interface as mentioned in Part 1.

Most models for multiphase flows in a porous material are still inaccurate to the zero-th order of the length ratio $\bar{\phi}/L$ as shown in the example in Part 1. The model accurate to first order of this length ratio has not been studied. Such model is essential to study the solid deformation caused by the surface tension. When the related integrals on the phase interfaces and contact lines are calculated correctly to the first order in the length ratio, the equations derived in the present paper is accurate to the order of the length scale ratio; and this averaged equation system provides unique opportunity to study the first order interface effects on multiphase flows in a porous solid. This is the main objective of the following section.

4. Flows with small interfaces

In this section, we study an approach to calculate probability integrals $\int \mathbf{s}_{12} \delta_{12} d\mathcal{P}$ on the fluid interfaces and $\int \Gamma_{12} \delta_c \mathbf{t}_{12} d\mathcal{P}$ on contact lines based on the physical meanings of the integrals. We assume that the typical interface size, L_f , is small compared to the macroscopic length scale, L , of the system. Such systems include multiphase flows in many porous materials commonly found in nature and used in industrial applications, but exclude multiphase flows in many fabric materials where fluids occupy the region outside of fibers, such as the flow of paint when a paint brush is dipped but not submerged into a pan of paint. In this case, the interfaces between paint and air along the cross section of the paint brush may extend along the entire brush width.

The probability interface integral $\int \mathbf{s}_{12} \delta_{12} d\mathcal{P}$ on the fluid interface integrates over all possible flows in the ensemble with the

interfaces touching the point of interest, \mathbf{x} , at time t , instead of integrating over an interface in a fixed flow. To calculate this integral, we first calculate the contribution from all interfaces with a specified shape S , and then average over all possible interface shapes. Let \mathbf{x}_c be the geometric center of an interface with shape S that touches point \mathbf{x} at time t . For a fixed shape, the interface location is uniquely determined by its geometric center; therefore the probability interface integral can be calculated as

$$\int \mathbf{s}_{12} \delta_{12} d\mathcal{P} = \overline{\int_S n_s^{(12)}(\mathbf{x}_c, t, S) \mathbf{s}_{12} dS(\mathbf{x}_c)}, \quad (26)$$

where $n_s^{(12)}(\mathbf{x}_c, t, S)$ is the number density of the interfaces with shape S at geometric center \mathbf{x}_c at time t , and the overline with superscript S denotes the average over all possible shapes. The integral integrates over all possible geometric centers of the interfaces that have shape S and pass point \mathbf{x} . For a specified shape S , the value of $n_s^{(12)} \mathbf{s}_{12}$ at \mathbf{x} is a function of the interface location represented by the geometric center \mathbf{x}_c , and the relative distance $\mathbf{r} = \mathbf{x} - \mathbf{x}_c$. We denote this dependency by $(n_s^{(12)} \mathbf{s}_{12})(\mathbf{r}, \mathbf{x}_c)$. For a constant surface tension coefficient Γ_{12} , \mathbf{s}_{12} is uniquely determined by the relative position \mathbf{r} and the shape S of the interface. There are cases in which the surface tension coefficient Γ_{12} depends on other flow parameters, such as temperature or chemical concentrations. To avoid the mathematical complications associated with these cases, in this section, we regard the surface tension coefficient Γ_{12} as the value averaged over these additional parameters.

The distance from the geometric center to a point on the interface is of the order of interface size. Under the small fluid interface assumption, with $\mathbf{x}_c = \mathbf{x} - \mathbf{r}$, a Taylor expansion in \mathbf{x}_c in $(n_s^{(12)} \mathbf{s}_{12})$ gives

$$(n_s^{(12)} \mathbf{s}_{12})(\mathbf{r}, \mathbf{x}_c) = (n_s^{(12)} \mathbf{s}_{12})(\mathbf{r}, \mathbf{x}) - \nabla [(n_s^{(12)} \mathbf{s}_{12})(\mathbf{r}, \mathbf{x})] \cdot \mathbf{r} + O(L_f^2/L^2). \quad (27)$$

The right-hand side of this relation is evaluated at the interface with the geometric center at \mathbf{x} instead of \mathbf{x}_c . By changing the integration variable from \mathbf{x}_c to \mathbf{r} and substituting (27) into (26), we have

$$\int \mathbf{s}_{12} \delta_{12} d\mathcal{P} = \mathbf{F}_s^{(12)} + \nabla \cdot \mathbf{T}_s^{(12)}, \quad (28)$$

where

$$\mathbf{F}_s^{(12)}(\mathbf{x}, t) = \overline{\int_S (n_s^{(12)} \mathbf{s}_{12})(\mathbf{r}, \mathbf{x}) dS(\mathbf{r})}, \quad (29)$$

and

$$\mathbf{T}_s^{(12)}(\mathbf{x}, t) = - \overline{\int_S (n_s^{(12)} \mathbf{s}_{12})(\mathbf{r}, \mathbf{x}) \otimes \mathbf{r} dS(\mathbf{r})} + O(L_f^2 \mathbf{F}_s^{(12)} / L). \quad (30)$$

The term $O(L_f^2 \mathbf{F}_s^{(12)} / L)$ in (30) accounts for the error in the Taylor expansion. The number density $n_s^{(12)}$ of interfaces with fixed shape S used in this derivation only exists conceptually and cannot be measured practically because of the number of parameters needed to describe a shape. To unequivocally study this system, we need a tool to study systems with infinite degrees of freedom. This tool is described in Appendix A. To avoid this difficulty, we now introduce the number density, $n^{(12)}$, of interfaces regardless of the shapes. This number density can be measured or calculated, at least in principle, in a practical system. Conceptually, this number density can be obtained by summing $n_s^{(12)}$ over all possible shapes, $n^{(12)}(\mathbf{x}, t) = \sum_S n_s^{(12)}(\mathbf{x}, t, S)$. With this number density of interfaces, for a shape dependent interface quantity $q(\mathbf{x}, t, S)$, we can define its average per interface as $\bar{q}(\mathbf{x}, t) = \overline{n_s^{(12)} q(\mathbf{x}, t, S)} / n^{(12)}(\mathbf{x}, t)$. In this way, (29) and (30) can be written as

$$\mathbf{F}_s^{(12)}(\mathbf{x}, t) = n^{(12)}(\mathbf{x}, t) \overline{\int_S \mathbf{s}_{12} dS}, \quad (31)$$

and

$$\mathbf{T}_s^{(12)}(\mathbf{x}, t) = -n^{(12)}(\mathbf{x}, t) \overline{\int_S \mathbf{s}_{12} \otimes \mathbf{r} dS} + O(L_f^2 \mathbf{F}_s^{(12)} / L). \quad (32)$$

The overlines in (31) and (32) denote the average values of the integrals per interface.

Similarly, for the probability integral on the contact line we have

$$\int \Gamma_{12} \delta_c \mathbf{t}_{12} d\mathcal{P} = \mathbf{F}_\ell^{(12)} + \nabla \cdot \mathbf{T}_\ell^{(12)}, \quad (33)$$

where $\mathbf{F}_\ell^{(12)}$ and $\mathbf{T}_\ell^{(12)}$ are the force and the stress on the solid phase resulting from the surface tension force applied by fluid interfaces. They can be calculated as

$$\mathbf{F}_\ell^{(12)}(\mathbf{x}, t) = n^{(12)}(\mathbf{x}, t) \overline{\int_{\partial S} \Gamma_{12} \mathbf{t}_{12} dC}, \quad (34)$$

and

$$\mathbf{T}_\ell^{(12)}(\mathbf{x}, t) = -n^{(12)}(\mathbf{x}, t) \overline{\int_{\partial S} \Gamma_{12} \mathbf{t}_{12} \otimes \mathbf{r} dC} + O(L_f^2 \mathbf{F}_\ell^{(12)} / L), \quad (35)$$

where ∂S is the contact line on the solid surface, and \mathbf{r} is the distance vector from the geometric center \mathbf{x} of the fluid interface S to the line element dC on the contact line.

This method of deriving (28) and (33) illustrates the physics of length scale separation in multiphase flows in a porous material. These derivations can be made mathematically rigorous as in Appendix A. Relations (28), (31) and (32) can be obtained by setting $\mathbf{f} = \mathbf{s}_{12}$ in (A.2) and using (A.8), (A.23) and (A.24). Relations (33)–(35) can be obtained by setting $\mathbf{f} = \Gamma_{12} \mathbf{t}_{12}$ in (A.17) and using (A.18), (A.25) and (A.26).

Relations (28) and (33) are extensions of the small particle approximation used by Zhang and Prosperetti (1994, 1997) for disperse two-phase flows. With this small interface approximation, the closure quantities $\mathbf{F}_\ell^{(12)}$, $\mathbf{T}_\ell^{(12)}$, $\mathbf{F}_s^{(12)}$ and $\mathbf{T}_s^{(12)}$ are calculated by integrating over a region (a surface or a curve), in a flow in the ensemble first, and then averaging over all possible flows that have these regions in the specified location. The integration of these regions implies length scales of the flows. These length scales are intrinsic to the physical system, and are not preset as in volume averaging methods. The use of the small interface approximation recovers the information of length scales obscured in an averaging process, and leads to the identification of the stresses $\mathbf{T}_\ell^{(12)}$ and $\mathbf{T}_s^{(12)}$ in the system. Without a tool similar to the small interface approximation or small particle approximation, volume averaging methods have been shown to miss many interesting terms for disperse two-phase flows (Marchioro et al., 2001a,b; Prosperetti, 2004). For multiphase flows in a porous medium, the stresses $\mathbf{T}_\ell^{(12)}$ and $\mathbf{T}_s^{(12)}$ are important in the mixture momentum equation, as we now show.

By summing the interface force \mathbf{f}_i , ($i = 1, 2, s$), as calculated in (8) and (14), over all the phases in the system and using (28) and (33), we find

$$\sum_i \mathbf{f}_i = \sum_i \langle \sigma_i \rangle \nabla \theta_i + \mathbf{F}_s^{(12)} + \mathbf{F}_\ell^{(12)} + \nabla \cdot (\mathbf{T}_s^{(12)} + \mathbf{T}_\ell^{(12)}). \quad (36)$$

Since the thickness of the phase interface is negligible, and on the edge of an interface there is neither a shear force in the direction normal to the interface, nor a force parallel to the edge, without an external force, the force balance on the interface leads to (Brackbill et al., 1992),

$$\int_S \mathbf{s}_{ij} dS + \oint_{\partial S} \Gamma_{ij} \mathbf{t}_{ij} dC = \mathbf{0}. \quad (37)$$

In systems with two fluid phases and a solid phase, an interface between fluids is either closed, such as a bubble, or bounded by a contact line on the solid surface. If the interface is closed, then ∂S is an empty set and according to (37) the surface integral is zero. This is in confirmation with the conclusion that surface tension does not produce net forces in bubbles (Prosperetti and Jones, 1984; Hesla et al., 1993). Using (31), (34) and (37), we find $\mathbf{F}_s^{(12)} + \mathbf{F}_\ell^{(12)} = \mathbf{0}$ in (36); and the mixture momentum equation becomes

$$\begin{aligned} \frac{\partial}{\partial t} (\rho_M \langle \mathbf{u}_M \rangle) + \nabla \cdot (\rho_M \langle \mathbf{u}_M \rangle \langle \mathbf{u}_M \rangle) \\ = \nabla \cdot \boldsymbol{\sigma}_M + \nabla \cdot (\mathbf{T}_s^{(12)} + \mathbf{T}_\ell^{(12)}) \\ + \nabla \cdot \sum_i [(\theta_i \sigma_i^{Re}) - \rho_i (\langle \mathbf{u}_i \rangle - \langle \mathbf{u}_M \rangle) (\langle \mathbf{u}_i \rangle - \langle \mathbf{u}_M \rangle)] + \rho_M \mathbf{g}, \end{aligned} \quad (38)$$

after summing momentum equation (4) over all phases, where $\rho_M = \sum_i \rho_i$ is the mixture density, $\boldsymbol{\sigma}_M = \sum_i \theta_i \langle \sigma_i \rangle$ is the mixture stress, and $\langle \mathbf{u} \rangle_M = \sum_i \rho_i \langle \mathbf{u}_i \rangle / \rho_M$ is the mixture velocity. This confirms the general conclusion that the phase interaction forces in a multiphase system do not usually cancel but produce stresses in the mixture equation and the mixture equation can be written in a conservative form (Zhang et al., 2006), except for the body force term.

Since $\mathbf{F}_s^{(12)} + \mathbf{F}_\ell^{(12)} = \mathbf{0}$, by using (28) and (33), the two integrals in (18) do not cancel each other exactly, but sum to $\nabla \cdot (\mathbf{T}_s^{(12)} + \mathbf{T}_\ell^{(12)})$. By comparing (31) with (32) we find that the ratio between the magnitudes of $\nabla \cdot \mathbf{T}_s^{(12)}$ and $\mathbf{F}_s^{(12)}$ is $O(L_f/L)$. The same is true between $\nabla \cdot \mathbf{T}_\ell^{(12)}$ and $\mathbf{F}_\ell^{(12)}$. Therefore the stresses are not present in the momentum equations with the zero-th order of accuracy in L_f/L . This explains the absence of these stresses in (25) in the previous section.

For the bundle-of-tubes model, we can calculate the two stresses directly using (32) and (35) to find

$$\mathbf{T}_s^{(12)} = \frac{\pi n^{(12)} \Gamma_{12} \phi^2(\mathbf{x}, t)}{12(1 + \sin \beta)} \text{diag} \left\{ (1 - \sin \beta)^2, 2[2 - \sin \beta(1 + \sin \beta)], \right. \\ \left. 2[2 - \sin \beta(1 + \sin \beta)] \right\}, \quad (39)$$

$$\mathbf{T}_\ell^{(12)} = \frac{\pi n^{(12)} \Gamma_{12} \phi^2(\mathbf{x}, t)}{4} \text{diag} \{ 1 - \sin \beta, \sin \beta, \sin \beta \}, \quad (40)$$

where $\text{diag}\{\dots\}$ denotes the diagonal tensor. For the example in the previous section, although it is not consistent to include these stresses in the momentum equations for the individual phases because of omission of terms of order L_f/L in the calculation of the average pressure as mentioned before, these stresses can be used in (38). Since both stresses are tensile stresses, Eq. (38) implies that a compressive mixture stress is needed to balance these stresses if the porous material has stress free boundaries. In other words, these stresses cause contraction of the porous material with stress-free boundaries, such as a drying sponge on a table.

5. Conclusions

Using ensemble phase averaging techniques for continuous multiphase flows, averaged momentum equations for multiphase flow in a porous solid are studied. After neglecting inertia for viscous multiphase flows, the averaged momentum equations for fluids are found to be in a form similar to Darcy's law but with additional terms representing the effect of phase interactions on

interfaces. These additional terms need to be modeled. In this study, the momentum equation for the solid phase is also derived.

Although for practical porous materials, much more work is needed on the closure relations for the averaged equations obtained in the present paper, using the bundle-of-tubes model for a porous material, the additional force terms identified in this work are shown to be important corrections needed for the models based on the straightforward extensions of Darcy's law, and for the models based on diffusions of fluids. In the momentum equation of the mixture, the stresses identified in this paper are shown to cause solid deformations during the drying and imbibition of fluids. However, the importance of the stresses has not been demonstrated in the momentum equations for individual phases because of the lack of needed accuracy in the approximate solution for the bundle-of-tubes model.

In this ensemble phase averaging method, with the small interface approximation, the closure relations are expressed as probability integrals of interactions on phase interfaces. These expressions can be used to unambiguously calculate the closure quantities needed in the ensemble phase averaged equations, when the information about phase interactions at the pore size scale is available. This method can potentially be used to provide new insight into how to construct closure relations using large amounts of data obtained from direct numerical simulations on more complicated pore morphologies.

Acknowledgments

The author would like to acknowledge Dr. Dali Yang. Her pioneering work of practical importance initiated my attention to the fundamental issues discussed in the present paper. The author also likes to acknowledge Dr. Rick M. Rauenzahn for many in-depth and constructive discussions.

Appendix A. Interface integrals

For a location \mathbf{x} , the integrals in (14) are calculated by integrating over all the flows that have an interface touching point \mathbf{x} , instead of integrating over an interface in a flow. To unequivocally calculate this type of integrals we need to study the probability of the flows.

We introduce a probability \mathcal{P} defined on a collection of subsets in the ensemble of the flows (Ash, 1972). A subset in the collection is called an event in probability theory. The probability is a set function that maps an event (a subset in the collection) to a real number between 0 and 1. We categorize flow systems into relative configuration classes. In each such class \mathcal{R} , the flows differ by only a rigid translation. For any given time, each flow \mathcal{F} in the ensemble belongs to one and only one such class. In this way a flow can be uniquely described by specifying the class \mathcal{R} that the flow belongs to and the position \mathbf{x}_R of a reference point in the configuration ($\mathcal{F} = \{\mathbf{x}_R, \mathcal{R}, t\}$). The probability of finding a system in the vicinity of a flow $\mathcal{F} = \{\mathbf{x}_R, \mathcal{R}, t\}$ can be calculated as $d\mathcal{P}(\mathcal{F}) = P_R(\mathbf{x}_R, t|\mathcal{R}) d^3\mathbf{x}_R d\mathcal{P}(\mathcal{R}, t)$, where $d\mathcal{P}(\mathcal{R}, t)$ is the probability of finding a flow that has a relative configuration in the vicinity of \mathcal{R} , and $P_R(\mathbf{x}_R, t|\mathcal{R})$ is the probability density function of finding the reference point of the configuration \mathcal{F} in the vicinity of \mathbf{x}_R . This relation among the probability $\mathcal{P}(\mathcal{F})$, the probability density $P_R(\mathbf{x}_R, t|\mathcal{R})$, and the probability $\mathcal{P}(\mathcal{R}, t)$ is a statement of Fubini's theorem (Billingsley, 1995). For this relation to hold the related probability spaces and the event sets are required to satisfy certain mathematical properties, which are met by most physical systems. For details regarding these requirements, readers are

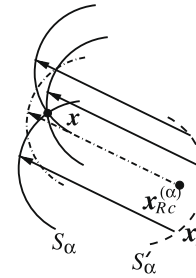


Fig. 3. Illustration of the relations in a relative configuration. For a specified point \mathbf{x} and a phase interface S_α , there is a surface S'_α formed by all reference points $\{\mathbf{x}'_R\}$ such that \mathbf{x} is on surface S_α in flow $\mathcal{F} = \{\mathbf{x}_R, \mathcal{R}, t\}$ when \mathbf{x}_R is on S'_α . Let \mathbf{x}'_{Rc} be the geometric center of surface S'_α . For the flow with the reference point \mathbf{x}_R located at the geometric center \mathbf{x}'_{Rc} of S'_α , location \mathbf{x} is the geometric center of the interface S_α in the flow.

referred to text books about measure on product space and Fubini's theorem (Billingsley, 1995).

For a specified relative configuration \mathcal{R} , there are only a finite number, N_R , of interfaces between phases i and j . Let $S_\alpha(\mathbf{x} - \mathbf{x}_R) = 0$ be the equation describing interface α ($1 \leq \alpha \leq N_R$) between phases i and j in the relative configuration \mathcal{R} . For a specified point \mathbf{x} in space, the point is on interface α in a flow \mathcal{F} with relative configuration \mathcal{R} , if and only if the reference point \mathbf{x}_R of the flow satisfies $S_\alpha(\mathbf{x} - \mathbf{x}_R) = 0$. In this way the position \mathbf{x} and surface α in the relative configuration \mathcal{R} define a surface S'_α consisting of all the reference points $\{\mathbf{x}'_R\}$, such that for a reference point \mathbf{x}_R on surface S'_α , the position \mathbf{x} is on interface α of flow $\mathcal{F} = \{\mathbf{x}_R, \mathcal{R}, t\}$. The relation among \mathbf{x} , interface S_α and surface S'_α is illustrated in Fig. 3.

We now introduce the geometrical center, \mathbf{x}'_{Rc} , of surface S'_α satisfying

$$\mathbf{x}'_{Rc} \int_{S_\alpha(\mathbf{x}-\mathbf{x}_R)=0} dS(\mathbf{x}_R) = \int_{S_\alpha(\mathbf{x}-\mathbf{x}_R)=0} \mathbf{x}_R dS(\mathbf{x}_R), \tag{A.1}$$

where the integration is over surface S'_α .

For a function $\mathbf{f}(\mathbf{x}, \mathbf{x}_R, \mathcal{R}; t)$, its interface integral can be calculated as

$$\begin{aligned} \mathbf{I} &= \int \mathbf{f}(\mathbf{x}, \mathbf{x}_R, \mathcal{R}; t) \delta_{ij} d\mathcal{P} \\ &= \int_R \sum_{\alpha=1}^{N_R} \int_{S_\alpha(\mathbf{x}-\mathbf{x}_R)=0} \mathbf{f}(\mathbf{x}, \mathbf{x}_R, \mathcal{R}; t) P_R(\mathbf{x}_R, t|\mathcal{R}) dS_\alpha(\mathbf{x}_R) d\mathcal{P}(\mathcal{R}, t). \end{aligned} \tag{A.2}$$

To calculate integral (A.2) we introduce a variable $\mathbf{y} = \mathbf{x} - \mathbf{x}_R$ and a function Φ as

$$\Phi(\mathbf{y}, \mathbf{x}_R, \mathcal{R}; t) = \mathbf{f}(\mathbf{x}_R + \mathbf{y}, \mathbf{x}_R, \mathcal{R}; t) P_R(\mathbf{x}_R, t|\mathcal{R}). \tag{A.3}$$

The variable \mathbf{y} represents the distance from the reference point \mathbf{x}_R to the point \mathbf{x} within the relative configuration \mathcal{R} . In many flows, the value of Φ is sensitive to a change in \mathbf{y} because this change represents a displacement of \mathbf{x} relative to important features, such as phase interfaces, in the flow. The second variable, \mathbf{x}_R , represents the translation of the configuration, while keeping the relative distances between \mathbf{x} and the important features unchanged. The change in the function Φ because of a change in \mathbf{x}_R is a consequence of the change in the environment that the flow is in. This change in the function value is often slower than the change caused by the relative motion between \mathbf{x} and the important features of the flow, especially when \mathbf{x} is in the vicinity of the important features, such as interfaces. In the form of Φ we distinguish variables \mathbf{y} that often causes a rapid change in the function value from the variable \mathbf{x}_R that usually causes a slow change in the function value in a multiphase flow. This distinction is similar to the

idea of separating physical quantities into the rapidly varying and slowly varying components as used by Prosperetti and Jones (1984). Using (A.3) we write (A.2) as

$$\mathbf{I} = \int_R \sum_{\alpha=1}^{N_R} \int_{S_\alpha(\mathbf{x}-\mathbf{x}_R)=0} \Phi(\mathbf{x}-\mathbf{x}_R, \mathbf{x}_R, \mathcal{R}; t) dS_\alpha(\mathbf{x}_R) d\mathcal{P}(\mathcal{R}, t). \quad (\text{A.4})$$

After changing variables

$$\mathbf{r}_\alpha = \mathbf{x}_{RC}^{(\alpha)} - \mathbf{x}_R \quad \text{and} \quad \mathbf{y}_c^{(\alpha)} = \mathbf{x} - \mathbf{x}_{RC}^{(\alpha)}, \quad (\text{A.5})$$

and noting $dS_\alpha(\mathbf{x}_R) = dS_\alpha(\mathbf{r}_\alpha)$, we can rewrite Eq. (A.4) as

$$\mathbf{I} = \int_R \sum_{\alpha=1}^{N_R} \int_{S_\alpha(\mathbf{y}_c^{(\alpha)}+\mathbf{r}_\alpha)=0} \Phi(\mathbf{y}_c^{(\alpha)} + \mathbf{r}_\alpha, \mathbf{x}_{RC}^{(\alpha)} - \mathbf{r}_\alpha, \mathcal{R}; t) dS_\alpha(\mathbf{r}_\alpha) d\mathcal{P}(\mathcal{R}, t). \quad (\text{A.6})$$

Applying Lagrange's mean value theorem to the second variable of Φ , one finds that there is a number η satisfying $0 < \eta < 1$, such that

$$\begin{aligned} \mathbf{I} = & \int_R \sum_{\alpha=1}^{N_R} \int_{S_\alpha(\mathbf{y}_c^{(\alpha)}+\mathbf{r}_\alpha)=0} \Phi(\mathbf{y}_c^{(\alpha)} + \mathbf{r}_\alpha, \mathbf{x}_{RC}^{(\alpha)}, \mathcal{R}; t) dS_\alpha(\mathbf{r}_\alpha) d\mathcal{P}(\mathcal{R}, t) \\ & - \int_R \sum_{\alpha=1}^{N_R} \int_{S_\alpha(\mathbf{y}_c^{(\alpha)}+\mathbf{r}_\alpha)=0} \mathbf{r}_\alpha \cdot \nabla_2 \Phi(\mathbf{y}_c^{(\alpha)} + \mathbf{r}_\alpha, \mathbf{x}_{RC}^{(\alpha)} - \eta \mathbf{r}_\alpha, \mathcal{R}; t) \\ & dS_\alpha(\mathbf{r}_\alpha) d\mathcal{P}(\mathcal{R}, t), \end{aligned} \quad (\text{A.7})$$

where ∇_2 denotes that the gradient is applied to the second variable of Φ . Using the second variable change in (A.5), we have $\nabla_2 \Phi(\mathbf{y}_c^{(\alpha)} + \mathbf{r}_\alpha, \mathbf{x}_{RC}^{(\alpha)} - \eta \mathbf{r}_\alpha, \mathcal{R}; t) = \nabla_x \Phi(\mathbf{y}_c^{(\alpha)} + \mathbf{r}_\alpha, \mathbf{x} - \mathbf{y}_c^{(\alpha)} - \eta \mathbf{r}_\alpha, \mathcal{R}; t)$. After the exchange of the order of ∇_x and the integrals, we find

$$\mathbf{I} = \mathbf{V} + \nabla_x \cdot \mathbf{T}, \quad (\text{A.8})$$

where

$$\mathbf{V}(\mathbf{x}, t) = \int_R \sum_{\alpha=1}^{N_R} \int_{S_\alpha(\mathbf{y}_c^{(\alpha)}+\mathbf{r}_\alpha)=0} \Phi(\mathbf{y}_c^{(\alpha)} + \mathbf{r}_\alpha, \mathbf{x} - \mathbf{y}_c^{(\alpha)}, \mathcal{R}; t) dS_\alpha(\mathbf{r}_\alpha) d\mathcal{P}(\mathcal{R}, t), \quad (\text{A.9})$$

$$\begin{aligned} \mathbf{T}(\mathbf{x}, t) = & - \int_R \sum_{\alpha=1}^{N_R} \int_{S_\alpha(\mathbf{y}_c^{(\alpha)}+\mathbf{r}_\alpha)=0} \Phi(\mathbf{y}_c^{(\alpha)} + \mathbf{r}_\alpha, \mathbf{x} - \mathbf{y}_c^{(\alpha)} - \eta \mathbf{r}_\alpha, \mathcal{R}; t) \\ & \otimes \mathbf{r}_\alpha dS_\alpha(\mathbf{r}_\alpha) d\mathcal{P}(\mathcal{R}, t). \end{aligned} \quad (\text{A.10})$$

For small interfaces, $|\eta \mathbf{r}_\alpha|$ is of the order of L_f . Using a Taylor expansion, we find

$$\begin{aligned} \mathbf{T}(\mathbf{x}, t) = & - \int_R \sum_{\alpha=1}^{N_R} \int_{S_\alpha(\mathbf{y}_c^{(\alpha)}+\mathbf{r}_\alpha)=0} \Phi(\mathbf{y}_c^{(\alpha)} + \mathbf{r}_\alpha, \mathbf{x} - \mathbf{y}_c^{(\alpha)}, \mathcal{R}; t) \\ & \otimes \mathbf{r}_\alpha dS_\alpha(\mathbf{r}_\alpha) d\mathcal{P}(\mathcal{R}, t) + O(\mathbf{V} L_f^2 / L). \end{aligned} \quad (\text{A.11})$$

From (A.3) and (A.5) we have

$$\mathbf{V}(\mathbf{x}, t) = \int_R \sum_{\alpha=1}^{N_R} P_R(\mathbf{x}_{RC}^{(\alpha)}, t | \mathcal{R}) \int_{S_\alpha(\mathbf{x}+\mathbf{r}_\alpha-\mathbf{x}_{RC}^{(\alpha)})=0} \mathbf{f}(\mathbf{x} + \mathbf{r}_\alpha, \mathbf{x}_{RC}^{(\alpha)}, \mathcal{R}; t) dS_\alpha(\mathbf{r}_\alpha) d\mathcal{P}(\mathcal{R}, t), \quad (\text{A.12})$$

and

$$\begin{aligned} \mathbf{T}(\mathbf{x}, t) = & - \int_R \sum_{\alpha=1}^{N_R} P_R(\mathbf{x}_{RC}^{(\alpha)}, t | \mathcal{R}) \int_{S_\alpha(\mathbf{x}+\mathbf{r}_\alpha-\mathbf{x}_{RC}^{(\alpha)})=0} \\ & \mathbf{f}(\mathbf{x} + \mathbf{r}_\alpha, \mathbf{x}_{RC}^{(\alpha)}, \mathcal{R}; t) \otimes \mathbf{r}_\alpha dS_\alpha(\mathbf{r}_\alpha) d\mathcal{P}(\mathcal{R}, t) + O(L_f^2 \mathbf{V} / L). \end{aligned} \quad (\text{A.13})$$

The surface integral over \mathbf{r}_α in (A.12) and (A.13) are performed in a fixed configuration with the reference point at $\mathbf{x}_{RC}^{(\alpha)}$. Furthermore, in this configuration point \mathbf{x} is the geometric center of the surface $S_\alpha(\mathbf{x} + \mathbf{r}_\alpha - \mathbf{x}_{RC}^{(\alpha)}) = 0$ as we now prove. Upon changing the integration variable from \mathbf{x}_R to \mathbf{r}_α using (A.5), we can rewrite (A.1) as

$$\mathbf{x}_{RC}^{(\alpha)} \int_{S_\alpha(\mathbf{x}+\mathbf{r}_\alpha-\mathbf{x}_{RC}^{(\alpha)})=0} dS_\alpha(\mathbf{r}_\alpha) = \int_{S_\alpha(\mathbf{x}+\mathbf{r}_\alpha-\mathbf{x}_{RC}^{(\alpha)})=0} (\mathbf{x}_{RC}^{(\alpha)} - \mathbf{r}_\alpha) dS_\alpha(\mathbf{r}_\alpha), \quad (\text{A.14})$$

or

$$\int_{S_\alpha(\mathbf{x}+\mathbf{r}_\alpha-\mathbf{x}_{RC}^{(\alpha)})=0} \mathbf{r}_\alpha dS_\alpha(\mathbf{r}_\alpha) = \mathbf{0}. \quad (\text{A.15})$$

This implies

$$\int_{S_\alpha(\mathbf{x}+\mathbf{r}_\alpha-\mathbf{x}_{RC}^{(\alpha)})=0} (\mathbf{x} + \mathbf{r}_\alpha) dS_\alpha(\mathbf{r}_\alpha) = \mathbf{x} \int_{S_\alpha(\mathbf{x}+\mathbf{r}_\alpha-\mathbf{x}_{RC}^{(\alpha)})=0} dS_\alpha(\mathbf{r}_\alpha); \quad (\text{A.16})$$

or \mathbf{x} is the geometric center of the interface $S_\alpha(\mathbf{x} + \mathbf{r}_\alpha - \mathbf{x}_{RC}^{(\alpha)}) = 0$ in the flow with the reference point at $\mathbf{x}_{RC}^{(\alpha)}$. Conversely, working backward, we can prove if \mathbf{x} is the geometric center of S_α , $\mathbf{x}_{RC}^{(\alpha)}$ is at the geometric center of S_α . The surface integrals in (A.12) and (A.13) integrate over surfaces with their geometric centers at \mathbf{x} . The probability integral averages over all such surface integrals according to the probability distribution of the ensemble.

The equations so far listed in this appendix are valid, if we redefine N_R above to be the number of curves defined in the relative configuration \mathcal{R} , and replace all the surface integrals by curve integrals along curves $C_\alpha(\mathbf{x} - \mathbf{x}_R) = 0$ ($\alpha = 1, \dots, N_R$). In this way the integrals in (A.1) used to define geometric center $\mathbf{x}_{RC}^{(\alpha)}$ also become curve integrals. If the curves are the edges of phase interfaces, in the derivation of corresponding relations for the curve integrals, we can choose that $\mathbf{x}_{RC}^{(\alpha)}$ remains to be the geometric center of the phase interface.

Corresponding to (A.8), a curve integral,

$$\mathbf{I}_c = \int_R \sum_{\alpha=1}^{N_R} \int_{C_\alpha(\mathbf{x}-\mathbf{x}_R)=0} \mathbf{f}(\mathbf{x}, \mathbf{x}_R, \mathcal{R}; t) P_R(\mathbf{x}_R, t | \mathcal{R}) dC_\alpha(\mathbf{x}_R) d\mathcal{P}(\mathcal{R}, t), \quad (\text{A.17})$$

can be calculated as

$$\mathbf{I}_c = \mathbf{V}_c + \nabla_x \cdot \mathbf{T}_c, \quad (\text{A.18})$$

where

$$\begin{aligned} \mathbf{V}_c(\mathbf{x}, t) = & \int_R \sum_{\alpha=1}^{N_R} P_R(\mathbf{x}_{RC}^{(\alpha)}, t | \mathcal{R}) \\ & \times \int_{C_\alpha(\mathbf{x}+\mathbf{r}_\alpha-\mathbf{x}_{RC}^{(\alpha)})=0} \mathbf{f}(\mathbf{x} + \mathbf{r}_\alpha, \mathbf{x}_{RC}^{(\alpha)}, \mathcal{R}; t) dC_\alpha(\mathbf{r}_\alpha) d\mathcal{P}(\mathcal{R}, t), \end{aligned} \quad (\text{A.19})$$

$$\begin{aligned} \mathbf{T}_c(\mathbf{x}, t) = & - \int_R \sum_{\alpha=1}^{N_R} P_R(\mathbf{x}_{RC}^{(\alpha)}, t | \mathcal{R}) \int_{C_\alpha(\mathbf{x}+\mathbf{r}_\alpha-\mathbf{x}_{RC}^{(\alpha)})=0} \mathbf{f}(\mathbf{x} + \mathbf{r}_\alpha, \mathbf{x}_{RC}^{(\alpha)}, \mathcal{R}; t) \\ & \otimes \mathbf{r}_\alpha dC_\alpha(\mathbf{r}_\alpha) d\mathcal{P}(\mathcal{R}, t) + O(L_f^2 \mathbf{V}_c / L), \end{aligned} \quad (\text{A.20})$$

and $\mathbf{x}_{RC}^{(\alpha)}$ is the reference point of the configuration such that in the configuration, position \mathbf{x} is either the geometric center of the curve or the geometric center of the surface bounded by the curve. Either choice for this center is permitted as long as the centers used to calculate \mathbf{V}_c and \mathbf{T}_c are the same. For simplicity, in the main text we use the geometric center of the interface and not the geometric center of the curve.

The number density $n_s(\mathbf{x}, t)$ of interfaces at point \mathbf{x} is defined as number of interfaces per unit volume with the geometric centers of the interfaces at \mathbf{x} . As proved before, \mathbf{x} is the geometric center of interface α if and only if \mathbf{x}_R is at $\mathbf{x}_{Rc}^{(\alpha)}$; therefore the number density of interfaces at point \mathbf{x} can be calculated as

$$\begin{aligned} n_s(\mathbf{x}, t) &= \int \sum_{\alpha=1}^{N_R} \delta(\mathbf{x}_{Rc}^{(\alpha)} - \mathbf{x}_R) P_R(\mathbf{x}_R, t | \mathcal{R}) d^3 x_R d\mathcal{P}(\mathcal{R}, t) \\ &= \int \sum_{\alpha=1}^{N_R} P_R(\mathbf{x}_{Rc}^{(\alpha)}, t | \mathcal{R}) d\mathcal{P}(\mathcal{R}, t). \end{aligned} \quad (\text{A.21})$$

For a quantity $g_s^s(\mathbf{x}, t, \mathcal{R})$ pertaining to an interface with the geometric center at \mathbf{x} , its average \bar{g}^s can be defined as

$$\begin{aligned} \bar{g}^s(\mathbf{x}, t) &= \frac{1}{n_s} \int \sum_{\alpha=1}^{N_R} \delta(\mathbf{x}_{Rc}^{(\alpha)} - \mathbf{x}_R) g_s^s(\mathbf{x}, t, \mathcal{R}) P_R(\mathbf{x}_R, t | \mathcal{R}) d^3 x_R d\mathcal{P}(\mathcal{R}, t) \\ &= \frac{1}{n_s} \int \sum_{\alpha=1}^{N_R} g_s^s(\mathbf{x}, t, \mathcal{R}) P_R(\mathbf{x}_{Rc}^{(\alpha)}, t | \mathcal{R}) d\mathcal{P}(\mathcal{R}, t). \end{aligned} \quad (\text{A.22})$$

By setting g^s to be the corresponding surface integrals in (A.12) and (A.13), we can write

$$\mathbf{V}(\mathbf{x}, t) = n_s(\mathbf{x}, t) \overline{\int_{S_x} \mathbf{f} dS_x}, \quad (\text{A.23})$$

and

$$\mathbf{T}(\mathbf{x}, t) = -n_s(\mathbf{x}, t) \overline{\int_{S_x} \mathbf{f} \otimes \mathbf{r}_\alpha dS_x} + O(L_f^2 \mathbf{V}/L), \quad (\text{A.24})$$

where S_x is the interface with its geometric center at \mathbf{x} .

Similarly, regarding the curve integrals in (A.19) and (A.20) as quantities pertaining to one of the surfaces bounded by the curve, we can write

$$\mathbf{V}_c(\mathbf{x}, t) = n_s(\mathbf{x}, t) \overline{\int_{C_x} \mathbf{f} dS_x}, \quad (\text{A.25})$$

and

$$\mathbf{T}_c(\mathbf{x}, t) = -n_s(\mathbf{x}, t) \overline{\int_{C_x} \mathbf{f} \otimes \mathbf{r}_\alpha dS_x} + O(L_f^2 \mathbf{V}_c/L). \quad (\text{A.26})$$

The tools developed in this appendix are extensions of the small particle approximation used by Zhang and Prosperetti (1994, 1997). In the small particle approximation, the surface S_x has to be closed as phase interfaces are in disperse two phase flows. This restriction is now removed and the surface S_x can be open. In this way the relations (A.8) and (A.18) can be used for multiphase flows in a porous medium, in which an interface between two phases is not necessarily closed and often ends on a surface of a third phase.

Appendix B. Extension to multiphase

In the main text we have used the ensemble phase averaging approach to derive the averaged equations for two-phase flows in a porous solid. The approach is not limited to two-phase flows and can be extended to multiphase flows in a porous solid. In this appendix we explain additions needed for such extension.

In the case of multiphase flows, a contact line on the surface of a phase may be shared by more than one interface as shown in Fig. 2. To account for this possibility, the force balance on a surface of phase i can be written as

$$\sigma_i \cdot \nabla C_i + \sum_{j \neq i} \sigma_j \cdot (\nabla C_j)_i + \sum_{j \neq i} \mathbf{s}_{ij} \delta_{ij} + \sum_{[k,m]} \sum_{\alpha=1}^{N_{km}} \Gamma_{km}^{(\alpha)} \mathbf{t}_{km}^{(\alpha)} \delta_c = \mathbf{0}, \quad (\text{B.1})$$

where δ_{ij} is the δ -function on interface between phases i and j , such that $(\nabla C_i)_j = \mathbf{n}_i \delta_{ij}$, with \mathbf{n}_i being the normal on the interface

toward phase i . In this extension to (9) and (10), the subscript $[k, m]$ denotes the summation over all possible phase pairs not involving phase i ($k \neq i$ and $m \neq i$) in the system, N_{km} is the total number of interfaces between phases k and m at the time in the flow, and $\delta_c = \delta_i \delta_m = \delta_i \delta_k$ is the δ -function on the contact line. Without external forces on the interface, the surface traction $\mathbf{s}_{ij} = \Gamma_{ij} \kappa_{ij} \mathbf{n}_i + \nabla_t \Gamma_{ij}$, where $\Gamma_{ij} = \Gamma_{ji}$ is the surface tension coefficient on the interface between phases i and j , $\kappa_{ij} = -\kappa_{ji}$ is the curvature of the interface, and $\nabla_t = \nabla - \mathbf{n}_i (\mathbf{n}_i \cdot \nabla)$ is the gradient operator tangent to the interface. The curvature κ_{ij} is positive if the interface is convex for phase i .

As we did to derive (11) and (12), we use (B.1) and average over flows in which \mathbf{x} is not on the contact lines at time t , to obtain

$$\mathbf{F}_i(\mathbf{x}, t) + \mathbf{F}_{ji}(\mathbf{x}, t) = (\langle \sigma_i \rangle - \langle \sigma_j \rangle) \cdot \int (\nabla C_i)_j d\mathcal{P} + \int \mathbf{s}_{ij} \delta_{ij} d\mathcal{P}, \quad (\text{B.2})$$

where \mathbf{F}_{ij} is defined the same as (7).

Since the contact lines have negligible mass, the forces on a contact line balance each other and produce zero net force. These forces include the forces in the last term of left-hand side of (B.1), the tangential component $\Gamma_{ij} \mathbf{t}_{ij} \delta_c$ contained in $\mathbf{s}_{ij} \delta_{ij}$, and the concentrated surface traction force $\sigma_s \cdot (\nabla C_s)_i$ from the solid phase on the contact line if the contact line is on the surface of the solid. Substituting (B.1) into (5), using the force balance on the contact line and (B.2), for fluid phase i we find a generalization of (8) as

$$\mathbf{f}_i = \sum_{j \neq i} \mathbf{F}_{ij}, \quad (\text{B.3})$$

where summation index j runs over all phases in the system, including the solid phase.

Similarly, by substituting (B.1) into (5) and using (B.2), for the solid phase, we find that (6) can be extended to give

$$\mathbf{f}_s = \sum_i \mathbf{F}_{si} + \sum_{[k,m]} \int \sum_{\alpha=1}^{N_{km}} \Gamma_{km}^{(\alpha)} \mathbf{t}_{km}^{(\alpha)} \delta_c d\mathcal{P}. \quad (\text{B.4})$$

The last term represents the surface tension force applied by phase interfaces on the contact lines on the solid surface. This term is not usually zero because the surface tension forces on a contact line on the solid surface do not usually balance each other, and a response from the solid phase is needed as shown in Fig. 1 and explained in Section 2.1.

By noting $(\nabla C_i)_j = -(\nabla C_j)_i$ and relation (13), we have the following identity:

$$\sum_{[i,j]} (\langle \sigma_i \rangle - \langle \sigma_j \rangle) \int (\nabla C_i)_j d\mathcal{P} = \sum_i \langle \sigma_i \rangle \sum_j \int (\nabla C_i)_j d\mathcal{P} = \sum_i \langle \sigma_i \rangle \nabla \theta_i. \quad (\text{B.5})$$

From (B.2)–(B.5) we have

$$\sum_i \mathbf{f}_i(\mathbf{x}, t) = \sum_i \langle \sigma_i \rangle \nabla \theta_i + \sum_{[i,j]} \int \mathbf{s}_{ij} \delta_{ij} d\mathcal{P} + \sum_{[i,j]} \int \sum_{\alpha=1}^{N_{ij}} \Gamma_{ij}^{(\alpha)} \mathbf{t}_{ij}^{(\alpha)} \delta_c d\mathcal{P}, \quad (\text{B.6})$$

where summation with subscript i is over all phases, summation with subscript $[i, j]$ is over all phase pairs; and the last term, from (B.4), represents the effect of contact lines on the solid surface. Since the surface tension on a solid surface is not well defined, as before, we treat the solid response as a stress in the solid phase and regard $\Gamma_{sj}^{(\alpha)} = \Gamma_{js}^{(\alpha)} = 0$, $\mathbf{s}_{sj} = \mathbf{s}_{js} = \mathbf{0}$ in (B.1), (B.2) and (B.6). The summation with subscript $[i, j]$ can then be regarded over fluid phase pairs only.

For a contact line not on a solid surface, we have

$$\sum_{[i,j]} \sum_{\alpha=1}^{N_{ij}} \Gamma_{ij}^{(\alpha)} \mathbf{t}_{ij}^{(\alpha)} = \mathbf{0}, \quad (\text{B.7})$$

because of the force balance on the contact line. Using (B.7), we find that (B.6) is still correct if we extend the summation in the last term of (B.6) to include all contact lines in the flow. In other words, the last term in (B.6) can be regarded as an integral over flows having edges of fluid interfaces touching \mathbf{x} at time t . By setting $\mathbf{f} = \mathbf{s}_{ij}$ in (A.2) and then using (A.8), (A.12) and (A.13) we have

$$\int \mathbf{s}_{ij} \delta_{ij} d\mathcal{P} = \mathbf{F}_s^{(ij)} + \nabla \cdot \mathbf{T}_s^{(ij)}, \tag{B.8}$$

where

$$\mathbf{F}_s^{(ij)} = \int_R \sum_{\alpha=1}^{N_{Rij}} P_R(\mathbf{x}_{Rc}^{(\alpha)}, t | \mathcal{R}) \int_{S_{\alpha}} \mathbf{s}_{ij} dS_{\alpha} d\mathcal{P}(\mathcal{R}, t), \tag{B.9}$$

and

$$\mathbf{T}_s^{(ij)} = - \int_R \sum_{\alpha=1}^{N_{Rij}} P_R(\mathbf{x}_{Rc}^{(\alpha)}, t | \mathcal{R}) \int_{S_{\alpha}} \mathbf{s}_{ij} \otimes \mathbf{r}_{\alpha} dS_{\alpha} d\mathcal{P}(\mathcal{R}, t) + O(L_f^2 \mathbf{F}_s^{(ij)} / L), \tag{B.10}$$

with N_{Rij} being the total number of fluid interfaces between phases i and j in the relative configuration \mathcal{R} . Since the last term in (B.6) can be regarded as a summation over all contact lines in a flow, the entire edge ∂S_{α} of any interface is included in the summation, including the parts of ∂S_{α} not on the surface of the solid phase but on a surface of a fluid phase. After setting $\mathbf{f} = \Gamma_{ij}^{(\alpha)} \mathbf{t}_{ij}^{(\alpha)}$ in (A.17) and then using (A.18)–(A.20), we have

$$\int \sum_{\alpha=1}^{N_{ij}} \Gamma_{ij}^{(\alpha)} \mathbf{t}_{ij}^{(\alpha)} \delta_c d\mathcal{P} = \mathbf{F}_{\ell}^{(ij)} + \nabla \cdot \mathbf{T}_{\ell}^{(ij)}, \tag{B.11}$$

where

$$\mathbf{F}_{\ell}^{(ij)}(\mathbf{x}, t) = \int_R \sum_{\alpha=1}^{N_{Rij}} P_R(\mathbf{x}_{Rc}^{(\alpha)}, t | \mathcal{R}) \int_{\partial S_{\alpha}} \Gamma_{ij}^{(\alpha)} \mathbf{t}_{ij}^{(\alpha)} dC_{\alpha} d\mathcal{P}(\mathcal{R}, t), \tag{B.12}$$

$$\begin{aligned} \mathbf{T}_{\ell}^{(ij)}(\mathbf{x}, t) = & - \int_R \sum_{\alpha=1}^{N_{Rij}} P_R(\mathbf{x}_{Rc}^{(\alpha)}, t | \mathcal{R}) \int_{\partial S_{\alpha}} \Gamma_{ij}^{(\alpha)} \mathbf{t}_{ij}^{(\alpha)} \\ & \otimes \mathbf{r}_{\alpha} dC_{\alpha} d\mathcal{P}(\mathcal{R}, t) + O(L_f^2 \mathbf{F}_{\ell}^{(ij)} / L). \end{aligned} \tag{B.13}$$

Using (37) we find $\mathbf{F}_s^{(ij)} + \mathbf{F}_{\ell}^{(ij)} = \mathbf{0}$, and (B.6) becomes

$$\sum_i \mathbf{f}_i(\mathbf{x}, t) = \sum_i (\sigma_i) \nabla \theta_i + \nabla \cdot \sum_{[ij]} (\mathbf{T}_s^{(ij)} + \mathbf{T}_{\ell}^{(ij)}). \tag{B.14}$$

With this relation, the mixture momentum equation can be written in a general conservative form similar to (38),

$$\begin{aligned} \frac{\partial}{\partial t} (\rho_M \langle \mathbf{u}_M \rangle) + \nabla \cdot (\rho_M \langle \mathbf{u}_M \rangle \langle \mathbf{u}_M \rangle) \\ = \nabla \cdot \boldsymbol{\sigma}_M + \nabla \cdot \sum_{[ij]} (\mathbf{T}_s^{(ij)} + \mathbf{T}_{\ell}^{(ij)}) \\ + \nabla \cdot \sum_i [(\theta_i \boldsymbol{\sigma}_i^{Re}) - \rho_i (\langle \mathbf{u}_i \rangle - \langle \mathbf{u}_M \rangle) (\langle \mathbf{u}_i \rangle - \langle \mathbf{u}_M \rangle)] + \rho_M \mathbf{g}. \end{aligned} \tag{B.15}$$

References

Ash, R.B., 1972. Real Analysis and Probability. Academic Press, Orlando, FL.

Batchelor, G.K., 1970. The stress system in a suspension of force-free particles. *J. Fluid Mech.* 41, 545–570.

Billingsley, P., 1995. Section 18. Product Measure and Fubini’s Theorem. *Probability and Measure*. Wiley, New York, NY.

Brackbill, J.U., Kothe, D.B., Zemach, C., 1992. A continuum method for modeling surface tension. *J. Comput. Phys.* 100, 335–354.

Breugem, W.P., Rees, D.A.S., 2006. A derivation of the volume-averaged Boussinesq equations for flow in porous media with viscous dissipation. *Transport Porous Med.* 61, 1–12.

Drew, D.A., Passman, S.L., 1999. *Theory of Multicomponent Fluids*. Springer, New York, NY (Chapter 9: Ensemble Averaging).

Dussan V, E.B., 1979. On the spreading of liquids on solid surface: statistical and dynamic contact lines. *Annu. Rev. Fluid Mech.* 11, 371–400.

Hall, C., 2007. Anomalous diffusion in unsaturated flow: fact or fiction? *Cement Concrete Res.* 37, 378–385.

Hassanizadeh, S.M., Gray, W.G., 1993. Thermodynamic basis of capillary pressure in porous media. *Water Resour. Res.* 29, 3389–3405.

Hesla, T.L., Huang, A.Y., Joseph, D.D., 1993. A note on the net force and moment on a drop due to surface forces. *J. Colloid Interf. Sci.* 158, 255–257.

Hilfer, R., 2006. Macroscopic capillarity without a constitutive capillary pressure function. *Physica A* 371, 209–225.

Hillel, D., 1980. *Fundamentals of Soil Physics*. Academic Press, San Diego, CA.

Küntz, M., Lavallée, P., 2001. Experimental evidence and theoretical analysis of anomalous diffusion during water infiltration in porous building materials. *J. Phys. D* 34, 2547–2554.

Landau, L.D., Lifshitz, E.M., 1987. *Fluid Dynamics*, second ed. Pergamon Press, Oxford (Chapter 7).

Leverett, M., 1941. Capillary behavior in porous solids. *Trans. Am. Inst. Min. Metall. Petrol. Eng.* 142, 152–168.

Liu, W., Peng, S.W., Mizukami, K., 1995. A general mathematical modeling for heat and mass transfer in unsaturated porous media: an application to free evaporative cooling. *Heat Mass Transfer* 31, 49–55.

Lockington, D.A., Parlange, J-Y., 2003. Anomalous water absorption in porous materials. *J. Phys. D* 36, 760–767.

Marchioro, M., Tanksley, M., Prosperetti, A., 2001a. Flow of spatially non-uniform suspensions. Part I: phenomenology. *Int. J. Multiphase Flow* 26, 783–831.

Marchioro, M., Tanksley, M., Wang, W., Prosperetti, A., 2001b. Flow of spatially non-uniform suspensions. Part II: systematic derivation of closure relations. *Int. J. Multiphase Flow* 27, 237–276.

Miller, C.A., Noegi, P., 1985. *Interfacial Phenomena*. Marcel Dekker, New York, NY.

Nordbotten, J.M., Celia, M.A., Dahle, K.H., Hassanizadeh, S.M., 2007. Interpretation of macroscale variables in Darcy’s law. *Water Resour. Res.* 43, W08430.

Nordbotten, J.M., Celia, M.A., Dahle, K.H., Hassanizadeh, S.M., 2008. On the definition of macroscale pressure for multiphase flow in porous media. *Water Resour. Res.* 44, W06S02.

Pachepsky, Y., Timlin, D., Rawls, W., 2003. Generalized Richards equation to simulate water transport in unsaturated soils. *J. Hydrol.* 272, 3–13.

Pedras, M.H.J., de Lemos, M.H.S., 2001. Macroscopic turbulence modeling for incompressible flow through undeformable porous media. *Int. J. Heat Mass Transfer* 44, 1081–1093.

Prosperetti, A., 2004. The average stress in incompressible disperse flow. *Int. J. Multiphase Flow* 30, 1011–1036.

Prosperetti, A., Jones, A.V., 1984. Pressure forces in disperse two-phase flows. *Int. J. Multiphase Flow* 10, 425–440.

Washburn, E.W., 1921. The dynamics of capillary flow. *Phys. Rev. 2nd series* 17, 273–283.

Yang, D., Currier, R.P., Zhang, D.Z., 2009. Ensemble phase averaged equations for multiphase flows in porous media Part 1: the bundle-of-tubes model. *Int. J. Multiphase Flow* 35, 628–639.

Zhang, D.Z., Prosperetti, A., 1994. Averaged equations for inviscid disperse two-phase flow. *J. Fluid Mech.* 267, 185–219.

Zhang, D.Z., Prosperetti, A., 1997. Momentum and energy equations for disperse two-phase flow and their closure for dilute suspensions. *Int. J. Multiphase Flow* 23, 425–453.

Zhang, D.Z., Ma, X., Rauenzahn, R.M., 2006. Inter-species stress in momentum equations for dense binary particulate systems. *Phys. Rev. Lett.* 97, 048301.

Zhang, D.Z., VanderHeyden, W.B., Zou, Q., Padial-Collins, N.T., 2007. Pressure calculation in disperse and continuous multiphase flows. *Int. J. Multiphase Flow* 33, 86–100.